

# THE DISTRIBUTION OF SMOOTH NUMBERS IN ARITHMETIC PROGRESSIONS

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## 1. INTRODUCTION

We say that a number  $n$  is  $y$ -smooth if all the prime factors of  $n$  lie below  $y$ . Let  $\mathcal{S}(y)$  denote the set of all  $y$ -smooth numbers, and let  $\mathcal{S}(x, y)$  denote the set of  $y$ -smooth numbers below  $x$ . Let  $\Psi(x, y)$  denote the number of smooth integers below  $x$ ; thus  $\Psi(x, y)$  is the cardinality of  $\mathcal{S}(x, y)$ . In this note we consider the distribution of smooth numbers among arithmetic progressions  $a \pmod{q}$ . We suppose that  $(a, q) = 1$ , and it is natural to expect that smooth numbers are equally distributed among such progressions: that is,

$$(1) \quad \Psi(x, y; q, a) := \sum_{\substack{n \in \mathcal{S}(x, y) \\ n \equiv a \pmod{q}}} 1 \sim \frac{1}{\phi(q)} \sum_{\substack{n \in \mathcal{S}(x, y) \\ (n, q) = 1}} 1 =: \frac{1}{\phi(q)} \Psi_q(x, y).$$

Naturally there are some limitations to when we may expect (1) to hold, but it seems safe to make the following conjecture.

**Conjecture I(A).** *Let  $A$  be a given positive real number. Let  $y$  and  $q$  be large with  $q \leq y^A$ . Then as  $\log x / \log q \rightarrow \infty$  we have*

$$\Psi(x, y; q, a) \sim \frac{1}{\phi(q)} \Psi_q(x, y).$$

In [5], [6] Granville established this Conjecture when  $A < 1$ . He noted that establishing the conjecture for arbitrarily large  $A$  would be difficult, since that would imply Vinogradov's conjecture that the least quadratic non-residue  $\pmod{p}$  lies below  $p^\epsilon$ . For, if  $p$  is a prime and  $y$  lies below the least quadratic non-residue  $\pmod{p}$  then all elements of  $\mathcal{S}(y)$  are quadratic residues  $\pmod{p}$ , and we cannot have the equidistribution property (1). The best known result towards Vinogradov's conjecture is that the least quadratic non-residue  $\pmod{p}$  lies below  $p^{1/4\sqrt{e}}$ . Thus it would be interesting to establish Conjecture I(A) for  $A < 4\sqrt{e}$ , and even more interesting to establish it for larger  $A$ . In this context, Harman [9] has shown that for  $q$  cube-free,  $q \leq y^{4\sqrt{e}-\epsilon}$  and  $q^{2+\epsilon} \leq x \leq q^{1/\epsilon}$  one has  $\Psi(x, y; q, a) \gg \Psi_q(x, y)/\phi(q)$ . A slightly weaker result holds for more general  $q$ ; see also the work of Balog and Pomerance [2] in this direction.

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**Theorem 1.** *Let  $y$  and  $q$  be large with  $q \leq y^{4\sqrt{e}-\epsilon}$ . For  $\exp(y^{1-\epsilon}) \geq x \geq y^{(\log \log y)^4}$  the asymptotic formula (1) holds.*

It seems plausible that with greater effort our methods could be extended to obtain Conjecture I when  $A < 4\sqrt{e}$ . We hope that an interested reader will accept that challenge.

As remarked above, there is a serious obstacle to establishing Conjecture I for any larger value of  $A$ . Namely, it may happen that the smooth numbers mostly lie in some subgroup of the group of reduced residues  $(\text{mod } q)$ ; for example, the subgroup of quadratic residues. However, within that subgroup we would expect equidistribution.

**Conjecture II(A).** *Let  $A$  be a given positive real number. Let  $y$  and  $q$  be large with  $q \leq y^A$ . There exists a constant  $C(A)$  depending only on  $A$ , and a subgroup  $H$  of  $(\mathbb{Z}/q\mathbb{Z})^*$  of index at most  $C(A)$  such that for any reduced residues  $a$  and  $b \pmod{q}$  with  $a/b \in H$  we have, as  $\log x / \log q \rightarrow \infty$*

$$(2) \quad \Psi(x, y; q, a) = \Psi(x, y; q, b) + o(\Psi_q(x, y)/\phi(q)).$$

Conjecture I is the stronger statement that  $H = (\mathbb{Z}/q\mathbb{Z})^*$ . We are optimistic that the methods developed here could be used to prove Conjecture II. Towards that end, we prove the following Theorem.

**Theorem 2.** *Let  $A$  be any positive real number and  $y$  and  $q$  be large with  $q \leq y^A$ . There exists a subgroup  $H$  of  $(\mathbb{Z}/q\mathbb{Z})^*$  of index bounded by  $C(A)$  such that for any two residue classes  $a$  and  $b$  with  $a/b \in H$  and all  $\exp(y^{1-\epsilon}) \geq x \geq y^{(\log \log y)^4}$  the asymptotic formula (2) holds.*

Let  $a \pmod{q}$  be an arithmetic progression with  $(a, q) = 1$ . Using the orthogonality of the characters  $\pmod{q}$  we may write

$$\Psi(x, y; q, a) = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)} \Psi(x, y; \chi),$$

where

$$(3) \quad \Psi(x, y; \chi) = \sum_{n \in \mathcal{S}(x, y)} \chi(n).$$

We expect that the main term arises from the principal character, and that the contribution of all other characters is negligible. This is indeed the case for the range of Theorem 1. In the range of Theorem 2 we shall establish that there are at most a bounded number (in terms of  $A$ ) of characters (of bounded order) for which the sum in (3) can be large. The subgroup  $H$  consists of those residue classes which take the value 1 on all these problem characters.

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## 2. PRELIMINARY OBSERVATIONS

It is convenient to introduce a smooth weight  $\Phi(x)$ . We suppose that  $\Phi : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$  is a function, smooth on that domain, and approximating the characteristic function of the interval  $[0, 1]$ . Concretely, given  $\epsilon > 0$  we shall take  $\Phi$  to be 1 on  $[0, 1 - \epsilon]$ , 0 on  $[1, \infty)$  so that  $\Phi$  approximates from below the characteristic function of  $[0, 1]$  or we shall take  $\Phi$  to be 1 on  $[0, 1]$  and 0 on  $[1 + \epsilon, \infty)$  getting an approximation from above. At the last step, we shall let  $\epsilon$  go to zero. With such a choice for  $\Phi$ , we shall consider

$$(2.1) \quad \Psi(x, y; q, a, \Phi) = \sum_{\substack{n \in \mathcal{S}(y) \\ n \equiv a \pmod{q}}} \Phi(n/x) = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)} \Psi(x, y; \chi, \Phi)$$

with

$$(2.2) \quad \Psi(x, y; \chi, \Phi) = \sum_{n \in \mathcal{S}(y)} \chi(n) \Phi(n/x).$$

We define for  $\operatorname{Re}(s) > 0$

$$L(s, \chi; y) = \prod_{p \leq y} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{n \in \mathcal{S}(y)} \frac{\chi(n)}{n^s}.$$

By Mellin inversion we note that, for any  $c > 0$ ,

$$(2.3) \quad \Psi(x, y; \chi, \Phi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, \chi; y) x^s \check{\Phi}(s) ds,$$

where

$$(2.4) \quad \check{\Phi}(s) = \int_0^\infty \Phi(t) t^{s-1} dt.$$

Repeated integration by parts shows that for  $\operatorname{Re}(s) > 0$ ,

$$\check{\Phi}(s) = - \int_0^\infty \Phi'(t) \frac{t^s}{s} dt = \int_0^\infty \Phi''(t) \frac{t^{s+1}}{s(s+1)} dt = \dots,$$

so that, for any integer  $k \geq 1$ ,

$$(2.5) \quad |\check{\Phi}(s)| \ll_{\Phi, k} \frac{1}{|s|(|s|+1) \cdots (|s|+k-1)}.$$

In practice, we shall need (2.5) only for some fixed large number  $k$ ; certainly  $k = 100$  will be sufficient.

Hildebrand and Tenenbaum [11] (see also the expository article [12]) developed the saddle point method to obtain an asymptotic for  $\Psi(x, y)$ . Their results give, with some

obvious modifications, an asymptotic formula for (2.3) in the case when  $\chi = \chi_0$  is the principal character. Let us begin by recalling some details of this result. In order to keep our argument transparent, we will assume throughout that  $\exp(y^{1-\epsilon}) \geq x \geq y^{(\log \log y)^4}$ . Moreover, since Granville's work applies when  $q \leq \sqrt{y}$  we assume from now on that  $\sqrt{y} \leq q \leq y^A$ . With more work we could relax these assumptions, and avoid the appeal to Granville's work.

In the Hildebrand-Tenenbaum argument, the line of integration in (2.3) is chosen carefully. They take  $c$  to be  $\alpha = \alpha(x, y)$  which is the unique solution to

$$(2.6) \quad \sum_{p \leq y} \frac{\chi_0(p) \log p}{p^\alpha - 1} = \log x.$$

As usual, we set  $u = (\log x)/\log y$ . For  $y \geq (\log x)^{1+\epsilon}$  we have (see Lemmas 1 and 2 of [11])

$$(2.7a) \quad \alpha(x, y) = 1 - \frac{\xi(u)}{\log y} + O\left(\frac{1}{\log x} + \frac{\log x}{y \log y}\right),$$

where  $\xi(u)$  is the unique solution to  $e^\xi = 1 + \xi u$  and it satisfies

$$(2.7b) \quad \xi(u) \sim \log(u \log u).$$

Note that in our range for  $x$  and  $y$ , we have that  $\alpha \gg \epsilon$  is bounded away from zero. With this choice for  $c$ , their asymptotic is

$$(2.8) \quad \Psi(x, y; \chi_0, \Phi) \sim \frac{x^\alpha L(\alpha, \chi_0; y) \check{\Phi}(\alpha)}{\sqrt{2\pi} \phi_2(\alpha, \chi_0; y)}$$

where, in our range of  $x$  and  $y$ ,

$$(2.9) \quad \phi_2(\alpha, \chi_0; y) = \sum_{p \leq y} \chi_0(p) \frac{p^\alpha}{(p^\alpha - 1)^2} \log^2 p \asymp \log x \log y.$$

We also record that in our range for  $x$  and  $y$  we have

$$(2.10) \quad \log L(\alpha, \chi_0; y) \sim u;$$

this follows by a simple partial summation argument.

Take  $c = \alpha$  in (2.3), and note that  $|L(\alpha + it, \chi; y)| \leq L(\alpha, \chi_0; y)$ . The rapid decay of  $\check{\Phi}(s)$  (see (2.5)) allows us to truncate the integral in (2.3):

$$(2.11) \quad \begin{aligned} \Psi(x, y; \chi, \Phi) &= \frac{1}{2\pi i} \int_{\alpha - i\sqrt{q}}^{\alpha + i\sqrt{q}} L(s, \chi; y) x^s \check{\Phi}(s) ds + O(L(\alpha, \chi_0) x^\alpha q^{-10}) \\ &= \frac{1}{2\pi i} \int_{\alpha - i\sqrt{q}}^{\alpha + i\sqrt{q}} L(s, \chi; y) x^s \check{\Phi}(s) ds + O(\Psi(x, y; \chi_0, \Phi) q^{-2}). \end{aligned}$$

To bound  $\Psi(x, y; \chi, \Phi)$  for non-principal characters, we divide the characters  $(\bmod q)$  into various sets based on the location of the zeros of  $L(s, \chi)$ . Let  $0 \leq j \leq (\log q)/2$  be an integer, and let  $\mathcal{R}_j(q)$  denote the region  $\{s : \operatorname{Re}(s) > 1 - j/\log q, |\operatorname{Im}(s)| \leq q\}$ . The set  $\Xi(j)$  consists of the non-principal characters  $\chi$  for which  $L(s, \chi)$  has no zeros in  $\mathcal{R}_j(q)$ , but has a zero in  $\mathcal{R}_{j+1}(q)$ . By the log-free zero density estimate (see, for example, Chapter 18 of Iwaniec and Kowalski [13]) we know that

$$(2.12) \quad |\Xi(j)| \leq C_1 e^{C_2 j},$$

for some absolute positive constants  $C_1$  and  $C_2$ .

There are three basic arguments used in the proof. If  $\chi \in \Xi(j)$  for some  $j \geq 10A \log \log q$ , a direct use of the implied zero-free region leads to a good bound for  $\Psi(x, y; \chi, \Phi)$ . This takes care of the vast majority of characters. Second, in the region where  $j \leq 10A \log \log q$  but  $j \geq 4A \log A + D$  for some absolute positive constant  $D$ , we use a Rodoskii type argument (see [15] and Chapter 9 of [14]) to bound  $\Psi(x, y; \chi, \Phi)$ . We are then left with a bounded number of problematic characters. We show that those problem characters have bounded order, and the subgroup  $H$  of Theorem 2 arises as the group of residue classes  $r$  with  $\chi(r) = 1$  for all these problem characters. Lastly when  $A < 4\sqrt{e}$ , Burgess's character sum estimates (see [3]) and reasoning along the lines of Vinogradov's  $\sqrt{e}$  argument lead to the treatment of problem characters, and thus to Theorem 1.

### 3. CONSEQUENCES OF A ZERO-FREE REGION: BASIC ARGUMENT

**Lemma 3.1.** *Let  $\chi \pmod{q}$  be a non-principal character with  $\chi \in \Xi(j)$  for some  $j \geq 0$ , and let  $|t| \leq q/2$ . Then for any  $z \geq 2$*

$$\sum_{n \leq z} \Lambda(n) \chi(n) n^{-it} \ll \frac{z(\log qz)^2}{q} + z^{1-j/\log q} (\log q)^2.$$

*Proof.* We may assume that  $j \geq 1$  else the bound is trivial. Therefore there are no issues with Siegel zeros. We follow a modification to the standard explicit formula argument (see for example Chapter 19 of Davenport [4]). That argument shows

$$\sum_{n \leq z} \Lambda(n) \chi(n) n^{-it} = - \sum_{|\gamma-t| \leq q/2} \frac{x^{\rho-it}}{\rho-it} + O(z(\log qz)^2/q) + O(z^{\frac{1}{2}}),$$

where  $\rho$  runs over the non-trivial zeros of  $L(s, \tilde{\chi})$  with  $\tilde{\chi}$  being the primitive character inducing  $\chi$ . Since  $\chi \in \Xi(j)$  we see that if  $\rho = \beta + i\gamma$  with  $|t - \gamma| \leq q/2$  then  $|\gamma| \leq q$  and so  $\beta \geq 1 - j/\log q$ . Using this, and splitting the sum over  $\gamma$  into intervals of length 1, and noting that each such interval has  $\ll \log q$  zeros, we obtain the Lemma.

**Lemma 3.2.** *Retain our assumptions on  $x, y$  and  $q$ . Suppose that  $j \geq 10A \log \log q$ , and that  $\chi$  is a non-principal character lying in  $\Xi(j)$ . Let  $\alpha = \alpha(x, y)$  be as in (2.6). Let  $B$  be a suitably large, but fixed positive number. For any  $\sigma \geq \alpha - Bj/\log x - 3 \log \log x / \log x$  and  $|t| \leq q/2$  we have*

$$\log |L(\sigma + it, \chi; y)| = o(u).$$

*Proof.* We split the primes below  $y$  into the small primes  $p \leq q^{5(\log \log q)/j}$  and the large primes  $q^{5(\log \log q)/j} < p \leq y$ . Note that  $q^{5(\log \log q)/j} \leq q^{1/(2A)} \leq \sqrt{y}$ , by our assumptions on  $j$ ,  $y$  and  $q$ . For the small primes we have

$$\begin{aligned} \sum_{p \leq q^{5(\log \log q)/j}} \log \left| 1 - \frac{\chi(p)}{p^{\sigma+it}} \right|^{-1} &\ll \sum_{p \leq q^{5(\log \log q)/j}} \frac{1}{p^{\alpha-Bj/\log x-3(\log \log x)/\log x}} \\ &\ll \sum_{p \leq q^{5(\log \log q)/j}} \frac{1}{p^{\alpha}} \ll \sum_{p \leq \sqrt{y}} \frac{1}{p^{\alpha}} \ll \sqrt{u}, \end{aligned}$$

using the prime number theorem, partial summation, and (2.7a,b).

Next we treat the large primes. Note that

$$\sum_{q^{5(\log \log q)/j} \leq p \leq y} \log \left| 1 - \frac{\chi(p)}{p^{\sigma+it}} \right|^{-1} = \sum_{q^{5(\log \log q)/j} \leq n \leq y} \operatorname{Re} \frac{\chi(n)\Lambda(n)}{n^{\sigma+it} \log n} + O\left(\sum_{p \leq y} \frac{1}{p^{2\sigma}}\right).$$

The error term above is easily seen to be  $o(u)$ . To handle the main term above we use partial summation together with Lemma 3.1. Put temporarily

$$S(z) = \sum_{q^{5(\log \log q)/j} \leq n \leq z} \Lambda(n)\chi(n)n^{-it}.$$

Partial summation gives

$$\begin{aligned} \sum_{q^{5(\log \log q)/j} \leq n \leq y} \operatorname{Re} \frac{\chi(n)\Lambda(n)}{n^{\sigma+it} \log n} &= \operatorname{Re} \int_{q^{5(\log \log q)/j}}^y \frac{1}{z^{\sigma} \log z} dS(z) \\ &\ll 1 + \frac{|S(y)|}{y^{\sigma} \log y} + \int_{q^{5(\log \log q)/j}}^y \frac{|S(z)|}{z^{\sigma+1} \log z} dz. \end{aligned}$$

Using Lemma 3.1 we may check that  $|S(z)|/z^{\sigma+1} \ll z^{-\alpha}(\log q)^{-3}$  in our range for  $z$ . Using this above, along with (2.7a,b) we conclude that

$$\sum_{q^{5(\log \log q)/j} \leq n \leq y} \operatorname{Re} \frac{\chi(n)\Lambda(n)}{n^{\sigma+it} \log n} \ll 1 + \frac{u}{\log q} = o(u).$$

We have established the desired bound for  $\log |L(\sigma + it, \chi; y)|$ .

Suppose  $\chi \in \Xi(j)$  for  $j \geq 10A \log \log q$ . With  $B$  suitably large, we will apply Lemma 3.2 and shift the contour of integration in (2.11) to the  $\alpha - Bj/\log x - 3 \log \log x/\log x$  line. By the rapid decay of  $\check{\Phi}(s)$  the horizontal line segments contribute an amount  $\ll \Psi(x, y; \chi_0, \Phi)q^{-2}$ . The remaining vertical line segment contributes, using Lemma 3.2,

$$\begin{aligned} &\ll \int_{\alpha-Bj/\log x-3 \log \log x/\log x-i\sqrt{q}}^{\alpha-Bj/\log x-3 \log \log x/\log x+i\sqrt{q}} |L(s, \chi; y)x^s \check{\Phi}(s)ds| \ll e^{o(u)} x^{\alpha} e^{-Bj} (\log x)^{-3} \\ &\ll \Psi(x, y; \chi_0, \Phi) e^{-Bj} (\log x)^{-2}, \end{aligned}$$

upon recalling (2.8), (2.9) and (2.10). Thus, for  $\chi \in \Xi(j)$  with  $j \geq 10A \log \log q$  we conclude that

$$(3.1) \quad \Psi(x, y; \chi, \Phi) \ll \Psi(x, y; \chi_0, \Phi) \left( \frac{e^{-Bj}}{(\log x)^2} + \frac{1}{q^2} \right).$$

Now we can explain what suitably large means for  $B$ : namely that  $B$  exceeds  $C_2$ , the constant appearing in the zero-density estimate (2.12). Choosing  $B$  that large, we conclude from (3.1) that

$$(3.2) \quad \sum_{j \geq 10A \log \log q} \sum_{\chi \in \Xi(j)} |\Psi(x, y; \chi, \Phi)| \ll \Psi(x, y; \chi_0, \Phi) \left( \frac{1}{(\log x)^2} + \frac{1}{q} \right).$$

This is our basic zero-density argument, and it takes care of all but  $\ll (\log q)^{10AC_2}$  characters  $\chi \pmod{q}$ .

#### 4. CONSEQUENCES OF A ZERO-FREE REGION: THE RODOSKIĀ ARGUMENT

There remain  $\ll (\log q)^{10AC_2}$  characters  $\chi \pmod{q}$  which are not covered by the argument of §3. We now give a second argument to prune this set of characters, leaving only a bounded number of characters left to be estimated.

**Proposition 4.1.** *Retain our ranges for  $x$ ,  $y$ , and  $q$ . There exists an absolute positive constant  $D$  such that if  $\chi \in \Xi(j)$  with  $j \geq 4A \log A + D$  then*

$$\Psi(x, y; \chi, \Phi) \ll \Psi(x, y; \chi_0, \Phi) \left( (\log x) e^{-\sqrt{u}/20} + q^{-2} \right).$$

We shall bound  $\Psi(x, y; \chi, \Phi)$  using (2.11). Using (2.5), we may express this bound as

$$(4.1) \quad \Psi(x, y; \chi, \Phi) \ll x^\alpha \max_{|t| \leq \sqrt{q}} |L(\alpha + it, \chi; y)| + q^{-2} \Psi(x, y; \chi_0, \Phi).$$

We now define

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 = \sum_{\substack{p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} \chi(p)p^{-it}}{p^\alpha}.$$

This is a distance function which satisfies a triangle inequality:

$$\mathbb{D}_\alpha(f_1, g_1; y) + \mathbb{D}_\alpha(f_2, g_2; y) \geq \mathbb{D}_\alpha(f_1 f_2, g_1 g_2; y),$$

where  $f_1, f_2, g_1, g_2$  are completely multiplicative functions taking values in the unit disc, and  $\mathbb{D}_\alpha(f, g; y)^2 = \sum_{p \leq y, p \nmid q} (1 - \operatorname{Re} \overline{f(p)}g(p))/p^\alpha$ . The triangle inequality above may be deduced easily from Cauchy-Schwarz; see also the paper [8] for a general discussion of such inequalities, and [1], [7] for some applications. Note that

$$|L(\alpha + it, \chi; y)| \ll |L(\alpha, \chi_0; y)| \exp \left( -\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \right),$$

and so from (4.1) (and recalling (2.8) and (2.9)) we obtain that

$$(4.2) \quad \begin{aligned} \Psi(x, y; \chi, \Phi) &\ll x^\alpha L(\alpha, \chi_0; y) \exp\left(-\min_{|t| \leq \sqrt{q}} \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2\right) + q^{-2} \Psi(x, y; \chi_0, \Phi) \\ &\ll \Psi(x, y; \chi_0, \Phi) \left(\frac{1}{q^2} + (\log x) \exp\left(-\min_{|t| \leq \sqrt{q}} \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2\right)\right). \end{aligned}$$

To proceed further we need some lower bounds on the distance function above; this is given in the following Lemma from which Proposition 4.1 is immediate.

**Lemma 4.2.** *Retain the notation of Proposition 4.1. If  $\chi \in \Xi(j)$  with  $j \geq 4A \log A + D$  then, for  $|t| \leq \sqrt{q}$  we have*

$$(4.3) \quad \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \geq \sqrt{u}/20.$$

The proof of Lemma 4.2 rests on some ideas of Rodoskiĭ [15]; we follow here the treatment given in Chapter 9 of Montgomery [14]. Observe that, using (2.7a,b),

$$(4.4) \quad \begin{aligned} \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 &\geq \sum_{\substack{\sqrt{y} \leq p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} \chi(p)p^{-it}}{p^\alpha} \geq \frac{y^{\frac{1-\alpha}{2}}}{\log y} \sum_{\substack{\sqrt{y} \leq p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} \chi(p)p^{-it}}{p} \log p \\ &\geq \frac{\sqrt{u}}{\log y} \sum_{\substack{\sqrt{y} \leq p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} \chi(p)p^{-it}}{p} \log p. \end{aligned}$$

Further let us define, as in Montgomery [14], the smooth weights

$$W(p) = \begin{cases} \log(p/\sqrt{y}) & \text{if } \sqrt{y} \leq p \leq y^{\frac{3}{4}} \\ \log(y/p) & \text{if } y^{\frac{3}{4}} \leq p \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any  $c > 0$

$$W(p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-w} \left( \frac{y^{w/2} - y^{w/4}}{w} \right)^2 dw.$$

From (4.4), we see that

$$(4.5) \quad \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \geq \frac{4\sqrt{u}}{\log^2 y} \sum_{p \nmid q} \frac{1 - \operatorname{Re} \chi(p)p^{-it}}{p} W(p) \log p.$$

Since

$$\sum_{p \nmid q} \frac{\log p}{p} W(p) \sim \frac{\log^2 y}{16},$$

we may conclude from (4.5) that

$$(4.6) \quad \mathbb{D}(1, \chi(p)p^{-it}; y)^2 \geq \frac{\sqrt{u}}{8} - \frac{4\sqrt{u}}{\log^2 y} \operatorname{Re} \sum_p \frac{\chi(p)}{p^{1+it}} W(p) \log p.$$

The desired bound (4.3) is a consequence of the following Lemma.



**Lemma 4.3.** *We keep the notations of Proposition 4.1. If  $\chi \in \Xi(j)$  with  $j \geq 4A \log A + D$  then, for  $|t| \leq \sqrt{q}$ , we have*

$$\operatorname{Re} \sum_p \frac{\chi(p)}{p^{1+it}} W(p) \log p \leq \frac{\log^2 y}{100}.$$

*Proof.* Let  $\tilde{\chi}$  denote the primitive character inducing the character  $\chi$ . Like  $L(s, \chi)$ , of course  $L(s, \tilde{\chi})$  is also free of zeros in the region  $\mathcal{R}_j(q)$ . If  $c > 0$  then

$$\sum_p \frac{\chi(p) \log p}{p^{1+it}} W(p) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{L'}{L}(1+it+w, \tilde{\chi}) \left( \frac{y^{w/2} - y^{w/4}}{w} \right)^2 dw + O(1).$$

Shifting contours to the left, this equals

$$- \sum_{\rho} \left( \frac{y^{(\rho-1-it)/2} - y^{(\rho-1-it)/4}}{\rho-1-it} \right)^2 + O(1),$$

where the sum is over all non-trivial zeros of  $L(s, \tilde{\chi})$ ; the contribution of the trivial zeros may be absorbed into the  $O(1)$  error term. The contribution of zeros  $\rho$  with  $|\operatorname{Im} \rho| > q$  is easily seen to be  $\ll (\log q)/q \ll 1$ . For a zero with  $|\operatorname{Im} \rho| \leq q$  we see by our hypothesis that the numerator of our sum is  $\ll y^{-j/(2 \log q)}$ . Thus we obtain that

$$(4.7) \quad \sum_p \frac{\chi(p) \log p}{p^{1+it}} W(p) \ll y^{-j/(2 \log q)} \sum_{|\operatorname{Im} \rho| \leq q} \frac{1}{|1+it-\rho|^2} + 1.$$

Now observe that if  $\rho = \beta + i\gamma$  with  $|\gamma| \leq y$ , and  $\beta \leq 1 - j/\log q$  then

$$\begin{aligned} \frac{1}{|1+it-\rho|^2} &\ll \frac{1}{|1+1/\log q + it - \rho|^2} = \frac{1}{1+1/\log q - \beta} \operatorname{Re} \frac{1}{1+1/\log q + it - \rho} \\ &\leq \frac{\log q}{j} \operatorname{Re} \frac{1}{1+1/\log q + it - \rho}. \end{aligned}$$

Thus the sum over zeros in (4.7) is

$$\ll \frac{\log q}{j} \sum_{\rho} \operatorname{Re} \frac{1}{1+1/\log q + it - \rho} \ll \frac{(\log q)^2}{j},$$

upon using a consequence of Hadamard factorization (see Davenport [4], chapter 12, equations (17) and (18)). Since  $y \geq q^{1/A}$  we deduce that

$$\sum_p \frac{\chi(p) \log p}{p^{1+it}} W(p) \ll e^{-j/(2A)} A^2 (\log y)^2 + 1.$$

Since  $j \geq 4A \log A + D$  for a suitably large  $D$ , this proves the Lemma.

There are  $\ll (\log q)^{10AC_2}$  characters  $\chi \pmod{q}$  lying in  $\Xi(j)$  for some  $10A \log \log q \geq j \geq 4A \log A + D$ . Therefore, by Proposition 4.1,

$$\sum_{j=4A \log A + D}^{10A \log \log q} \sum_{\chi \in \Xi(j)} |\Psi(x, y; \chi, \Phi)| \ll \Psi(x, y; \chi_0, \Phi) \left( (\log x)^{10AC_2+1} e^{-\sqrt{u}/20} + q^{-2} \right).$$

In our range for  $x$  and  $y$ , we have  $u \gg (\log \log x)^4$  and so we obtain that

$$(4.8) \quad \sum_{j=4A \log A + D}^{10A \log \log q} \sum_{\chi \in \Xi(j)} |\Psi(x, y; \chi, \Phi)| \ll \Psi(x, y; \chi_0, \Phi) \left( \frac{1}{(\log x)^2} + \frac{1}{q} \right).$$

## 5. A FEW PROBLEM CHARACTERS

In view of our work in §4, it remains only to consider characters  $\chi \pmod{q}$  with  $\chi \in \Xi(j)$  for some  $j \leq 4A \log A + D$ . By (2.12) there are only a bounded number  $B = B(A)$ , say, of such characters. We now define a set  $\mathcal{B}$  of *problem characters*. A non-principal character  $\chi$  belongs to this set  $\mathcal{B}$  precisely if it has order at most  $B$  and lies in  $\Xi(j)$  for some  $j \leq 4A \log A + D$ . With Theorem 2 in mind, we define  $H$  to be the subgroup of reduced residues  $h \pmod{q}$  with  $\chi(h) = 1$  for all  $\chi \in \mathcal{B}$ . Since  $\mathcal{B}$  contains at most  $B$  characters, all of order at most  $B$ , we see that the index of  $H$  in  $(\mathbb{Z}/q\mathbb{Z})^*$  is at most  $B^B$ .

**Proposition 5.1.** *Retain our ranges for  $x$ ,  $y$ , and  $q$ . If  $\chi$  is non-principal, and  $\chi \notin \mathcal{B}$  then*

$$\Psi(x, y; \chi, \Phi) \ll \Psi(x, y; \chi_0, \Phi) \left( \frac{1}{(\log x)^2} + \frac{1}{q} \right).$$

*Proof.* We first show that for all  $|t| \leq \sqrt{q}/(2B)$

$$(5.1) \quad \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \geq \frac{\sqrt{u}}{40B^2}.$$

If not, there exists  $t_\chi$  with  $|t_\chi| \leq \sqrt{q}/B$ , and with

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it_\chi}; y)^2 \leq \frac{\sqrt{u}}{40B^2}.$$

By the triangle inequality it follows that

$$\mathbb{D}_\alpha(1, \chi(p)^k p^{-ikt_\chi}; y)^2 \leq \frac{k^2 \sqrt{u}}{40B^2}.$$

By Lemma 4.2 we see that  $\chi, \chi^2, \dots, \chi^{B+1}$  must all be in  $\cup_{j \leq 4A \log A + D} \Xi(j)$ . Since there are at most  $B$  elements in  $\cup_{j \leq 4A \log A + D} \Xi(j)$ , it follows that two of the  $B+1$  characters listed above are the same. But then  $\chi$  would have order at most  $B$  and would be in  $\cup_{j \leq 4A \log A + D} \Xi(j)$ , contradicting our hypothesis that  $\chi \notin \mathcal{B}$ .

We now use (2.11), invoking (5.1) for  $|t| \leq \sqrt{q}/(2B)$  and the rapid decay of  $\check{\Phi}$  for  $\sqrt{q}/(2B) \leq |t| \leq \sqrt{q}$ . We conclude that

$$\begin{aligned} \Psi(x, y; \chi, \Phi) &\ll x^\alpha L(\alpha, \chi_0; y) \left( \exp(-\sqrt{u}/(40B^2)) + q^{-2} \right) \\ &\ll \Psi(x, y; \chi_0, \Phi) \left( (\log x) \exp(-\sqrt{u}/(40B^2)) + q^{-1} \right). \end{aligned}$$

Since  $u \gg (\log \log x)^4$  in our ranges for  $x$  and  $y$ , we obtain the Proposition.

Now we examine more closely the situation for characters of bounded order.

**Lemma 5.2.** *Let  $\chi$  be a character with order  $k > 1$ . In the range  $|kt| \leq 1/\log y$  we have*

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y) \gg \max \left( \mathbb{D}_\alpha(1, \chi(p); y)^2, |t|^2 \log x \log y \right),$$

while in the range  $1/\log y \leq |kt| \leq y$  we have

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y) \gg \frac{u}{k^2 \log^2 u}.$$

*Proof.* The triangle inequality gives

$$(5.2) \quad \mathbb{D}_\alpha(1, \chi(p)p^{-it}; y) \geq \frac{1}{k} \mathbb{D}_\alpha(1, p^{-ikt}; y).$$

Consider first the case  $y \geq |kt| \geq 1/\log y$ . Here we have

$$\mathbb{D}_\alpha(1, p^{-ikt}; y)^2 \geq \frac{1}{\log y} \sum_{\substack{p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} p^{-ikt}}{p^\alpha} \log p \gg \frac{1}{\log y} \left( \frac{y^{1-\alpha}}{1-\alpha} - \frac{y^{1-\alpha}}{|1-\alpha+ikt|} \right),$$

using the argument of the prime number theorem (using the Littlewood or Vinogradov zero-free regions for  $\zeta(s)$ ). Using (2.7) we conclude that

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \gg \frac{u}{k^2 \log^2 u}.$$

Using this in (5.2) we obtain our second assertion.

Now consider the range  $|kt| \leq 1/\log y$ . From (2.9) we obtain that

$$(5.3) \quad \sum_{\substack{p \leq y \\ p \nmid q}} \frac{1 - \operatorname{Re} p^{-ikt}}{p^\alpha} \asymp \sum_{\substack{p \leq y \\ p \nmid q}} \frac{(kt \log p)^2}{p^\alpha} \asymp (kt)^2 \log x \log y.$$

By (5.2) it follows that

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \gg t^2 \log x \log y,$$

which is one of the bounds in our first assertion. Moreover, by the triangle inequality we get that

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y) + \mathbb{D}_\alpha(1, p^{it}; y) \geq \mathbb{D}_\alpha(1, \chi(p); y).$$

As in (5.3) we see that if  $|t| \leq 1/\log y$  then

$$\mathbb{D}_\alpha(1, p^{it}; y)^2 \asymp t^2 \log x \log y.$$

Therefore

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 + O(t^2 \log x \log y) \gg \mathbb{D}_\alpha(1, \chi(p); y)^2,$$

and the other bound claimed in our first assertion follows.

**Lemma 5.3.** *Let  $\chi$  be a character of order  $1 < k \leq B$ , and let  $y \geq q^{\frac{1}{4\sqrt{e}}+\delta}$ . Then*

$$\mathbb{D}_\alpha(1, \chi; y)^2 \geq \frac{\delta}{4k} \log u + \log \delta + O(1).$$

*Proof.* Put  $z = y^{\sqrt{e}-\delta} > q^{\frac{1}{4}+\delta}$ . We define the completely multiplicative function  $f(n)$  by setting  $f(p) = 1$  for  $p \leq y$  and  $f(p) = \chi(p)$  for  $y < p \leq z$ . Since  $z < y^2$  we note that for  $n \leq z$  we have  $f(n) = 1 - \sum_{p|n} (1 - f(p))$ . Therefore

$$\begin{aligned} \operatorname{Re} \sum_{n \leq z} f(n) &= z - z \sum_{y \leq p \leq z} \operatorname{Re} \frac{1 - \chi(p)}{p} + o(z) \geq z \left( 1 - \sum_{y \leq p \leq z} \frac{2}{p} + o(1) \right) \\ &= z \left( 1 - 2 \log \frac{\log z}{\log y} + o(1) \right), \end{aligned}$$

and so

$$(5.4) \quad \left| \sum_{n \leq z} f(n) \right| \gg \delta z.$$

Now let us write  $f(n) = \sum_{d|n} g(d) \chi(n/d)$  where  $g$  is a multiplicative function with  $g(p) = 1 - \chi(p)$  for  $p \leq y$  and  $g(p) = 0$  for  $y < p \leq z$ . We see that

$$(5.5) \quad \sum_{n \leq z} f(n) = \sum_{d \leq z} g(d) \sum_{m \leq z/d} \chi(m).$$

For the terms  $d \leq y^{\delta/2}$ , so that  $z/d > q^{\frac{1}{4}+\frac{\delta}{2}}$ , we use Burgess's character sum estimates [3]. The refinement of Heath-Brown (see [10], Lemma 2.4) applies, since our character has bounded order. For such  $d$  we see that  $\sum_{m \leq z/d} \chi(m) \ll (z/d)/(\log z)^3$  say, and hence

$$\sum_{d \leq y^{\delta/2}} g(d) \sum_{m \leq z/d} \chi(m) \ll \frac{z}{(\log z)^3} \sum_{d \leq y^{\delta/2}} \frac{|g(d)|}{d} \ll \frac{z}{(\log z)^3} \sum_{d \leq z} \frac{d(d)}{d} = o(z).$$

The contribution of terms  $d > y^{\delta/2}$  to (5.5) is bounded in magnitude by

$$z \sum_{y^{\delta/2} \leq d \leq z} \frac{|g(d)|}{d} \leq \frac{z}{y^{\delta(1-\alpha)/2}} \sum_{d \leq z} \frac{|g(d)|}{d^\alpha} \ll \frac{z}{y^{\delta(1-\alpha)/2}} \exp \left( \sum_{p \leq y} \frac{|1 - \chi(p)|}{p^\alpha} \right).$$

Since  $\chi$  has order  $k$ , we have  $|1 - \chi(p)| \leq k(1 - \operatorname{Re} \chi(p))$ , and so the above is, using (2.7a,b),

$$\ll \frac{z}{u^{\delta/2}} \exp \left( k \mathbb{D}_\alpha(1, \chi; y)^2 \right).$$

From (5.4) and (5.5) we conclude that

$$\delta z \ll \left| \sum_{n \leq z} f(n) \right| \ll o(z) + \frac{z}{u^{\delta/2}} \exp \left( k \mathbb{D}_\alpha(1, \chi; y)^2 \right),$$

and our Lemma follows.

**Proposition 5.4.** *Retain our ranges for  $x$ ,  $y$ , and  $q$ . Let  $\chi \pmod{q}$  be a character of order  $1 < k \leq B$ . Then, for any  $1 \leq U \leq \sqrt{u}$  we have, for some positive constant  $c$ ,*

$$\begin{aligned} \Psi(x, y; \chi, \Phi) &= \frac{1}{2\pi} \int_{|t| \leq U/\sqrt{\log x \log y}} x^{\alpha+it} L(\alpha+it, \chi; y) \check{\Phi}(\alpha+it) dt \\ &\quad + O\left(\Psi(x, y; \chi_0, \Phi) \left( \frac{1}{q^2} + \frac{1}{(\log x)^2} + e^{-cU^2} \right)\right). \end{aligned}$$

If  $A < 4\sqrt{e} - 100\delta$  then for some small positive constant  $c$

$$\Psi(x, y; \chi, \Phi) \ll \Psi(x, y; \chi_0, \Phi) u^{-c\delta}.$$

*Proof.* We start with the expression (2.11). We split the integral over  $|\operatorname{Im} s| \leq \sqrt{q}$  into various ranges. The rapid decay of  $\check{\Phi}(s)$  shows that the contribution to the integral from  $y \leq |\operatorname{Im} s| \leq \sqrt{q}$  is  $\ll \Psi(x, y; \chi_0, \Phi) q^{-2}$ . In the range  $1/\log y \leq |\operatorname{Im} s| \leq y$  we use the second bound of Lemma 5.2. Thus the contribution of this range is  $\ll \Psi(x, y; \chi, \Phi)(\log x) \exp(-Cu/\log^2 u)$  for some constant  $C$ . Since  $u \gg (\log \log x)^4$  this contribution is  $\ll \Psi(x, y; \chi, \Phi)/(\log x)^2$ . In the range  $U/\sqrt{\log x \log y} \leq |\operatorname{Im} s| \leq 1/\log y$ , we use the first assertion of Lemma 5.2 which gives  $\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \gg |t|^2 \log x \log y$ . It follows that the contribution of this range is  $\ll x^\alpha L(\alpha, \chi_0; y) \exp(-cU^2)/\sqrt{\log x \log y} \ll \Psi(x, y; \chi_0, \Phi) e^{-cU^2}$  for some positive constant  $c$ . Piecing these statements together, we obtain the first assertion of the Proposition.

To prove the second assertion, we choose  $U = \sqrt{c^{-1} \log u}$ . From Lemmas 5.2 and 5.3 it follows that for  $|t| \leq U/\sqrt{\log x \log y}$

$$\mathbb{D}_\alpha(1, \chi(p)p^{-it}; y)^2 \gg \mathbb{D}_\alpha(1, \chi(p); y)^2 \gg \delta \log u.$$

Using this estimate to handle the integral in our first assertion, the Proposition follows.

## 6. PROOFS OF THE MAIN THEOREMS

*Proof of Theorem 1.* Combining (2.1), (3.2), (4.8), Proposition 5.1 and the second part of Proposition 5.4 we obtain that if  $A \leq 4\sqrt{e} - 100\delta$  then

$$\Psi(x, y; q, a, \Phi) = \frac{1}{\phi(q)} \Psi(x, y; \chi_0, \Phi) \left( 1 + O(u^{-c\delta}) \right),$$

for some positive constant  $c$ . We now take  $\Phi$  to be 1 on  $[0, 1 - \epsilon]$  and 0 on  $[1, \infty)$  to get a lower bound for  $\Psi(x, y; q, a)$ ; and  $\Phi$  to be 1 on  $[0, 1]$  and 0 on  $[1 + \epsilon, \infty)$  to get an upper bound for  $\Psi(x, y; q, a)$ . Taking  $\epsilon$  sufficiently small, we obtain Theorem 1.

*Proof of Theorem 2.* Combining (2.1), (3.2), (4.8) and Proposition 5.1 we obtain that (6.1)

$$\Psi(x, y; q, a, \Phi) = \frac{1}{\phi(q)} \Psi(x, y; \chi_0, \Phi) \left( 1 + O\left( \frac{1}{q} + \frac{1}{(\log x)^2} \right) \right) + \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{B}} \overline{\chi(a)} \Psi(x, y; \chi, \Phi).$$

Once again we take  $\Phi$  to be 1 on  $[0, 1 - \epsilon]$  and 0 on  $[1, \infty)$  to get a lower bound for  $\Psi(x, y; q, a)$ ; and  $\Phi$  to be 1 on  $[0, 1]$  and 0 on  $[1 + \epsilon, \infty)$  to get an upper bound for  $\Psi(x, y; q, a)$ . Note that in either case  $\check{\Phi}(\alpha + it) = 1/(\alpha + it) + O(\epsilon)$ . Therefore, from (6.1), (2.9) and Proposition 5.4 (taking there  $U = 1/\sqrt{\epsilon}$ ) we may conclude that (for large  $x, y$  and  $q$  lying in our ranges)

$$(6.2) \quad \begin{aligned} \Psi(x, y; q, a) &= \frac{1}{\phi(q)} \frac{x^\alpha L(\alpha, \chi_0; y)}{\sqrt{2\pi\phi_2(\alpha, \chi_0; y)}} + O\left(\frac{\sqrt{\epsilon}}{\phi(q)} \Psi(x, y; \chi_0)\right) \\ &+ \frac{1}{\phi(q)} \sum_{\chi \in \mathcal{B}} \overline{\chi(a)} \frac{1}{2\pi} \int_{|t| \leq 1/\sqrt{\epsilon \log x \log y}} x^{\alpha+it} L(\alpha + it, \chi; y) \frac{dt}{\alpha + it}. \end{aligned}$$

Recall that  $H$  is the subgroup of residues  $h$  such that  $\chi(h) = 1$  for all  $\chi \in \mathcal{B}$ , and that it has index at most  $B^B$ . If  $a/b \in H$  then  $\chi(a) = \chi(b)$  for each  $\chi \in \mathcal{B}$ . Therefore (6.2) gives identical expressions for both  $\Psi(x, y; q, a)$  and  $\Psi(x, y; q, b)$ . Consequently

$$\Psi(x, y; q, a) = \Psi(x, y; q, b) + O\left(\frac{\sqrt{\epsilon}}{\phi(q)} \Psi(x, y; \chi_0)\right).$$

This proves Theorem 2.

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